Journal of Computational and Applied Mathematics 234 (2010) 924–933

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Vector continuous-time programming without differentiability

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ARTICLE INFO

Article history:
Received 20 March 2009
Received in revised form 30 November 2009

MSC:
90C26
90C29
90C30
90C46

Keywords:
Continuous-time programming
Multiobjective programming
Preinvex functions
Scalarization
Saddle point optimality
Duality

ABSTRACT

In this work continuous-time programming problems of vector optimization are considered. Firstly, a nonconvex generalized Gordan's transposition theorem is obtained. Then, the relationship with the associated weighting scalar problem is studied and saddle point optimality results are established. A scalar dual problem is introduced and duality theorems are given. No differentiability assumption is imposed.

1. Introduction

The following vector continuous-time programming problem is addressed:

$$\begin{aligned} &\text{minimize} && \phi(x) = \int_0^T f(x(t), t)\mathrm{d}t \\ &\text{subject to} && g(x(t), t) \leq 0 \quad \text{a.e. in } [0, T], \\ &&& x \in X. \end{aligned} \tag{VCP}$$

Here X is a nonempty subset of the Banach space $L_\infty^n[0, T]$, $\phi : X \to \mathbb{R}^p$, $f(x(t), t) = \xi(x)(t)$, $g(x(t), t) = \gamma(x)(t)$, where $\xi : X \to \Lambda_1^p[0, T]$, and $\gamma : X \to \Lambda_1^m[0, T]$. $L_\infty^n[0, T]$ denotes the space of all n-dimensional vector valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0, T] \subset \mathbb{R}$, with norm $\|\cdot\|_\infty$ defined as

$$\|x\|_\infty = \max_{1\leq j\leq n} \operatorname{ess\,sup}\{|x_j(t)|,\ 0 \leq t \leq T\},$$

where for each $t \in [0, T]$, $x_j(t)$ is the j-th component of $x(t) \in \mathbb{R}^n$ and $\Lambda_1^m[0, T]$ denotes the space of all m-dimensional vector-valued functions which are essentially bounded and Lebesgue measurable, defined on $[0, T]$, with the norm $\|\cdot\|_1$ defined as

$$\|y\|_1 = \max_{1\leq j\leq m} \int_0^T |y_j(t)|\mathrm{d}t.$$

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doi:10.1016/j.cam.2010.02.012

Continuous-time problems were introduced in Bellman [1] in connection with production-inventory "bottleneck processes", where he considered a continuous-time linear programming problem. In 1965 Tyndall [2] generalized Bellman's initial formulations by giving mathematically rigorous treatment to the problem, in the linear case. Nonlinear problems were first investigated, in 1968 by Hanson [3] and Hanson and Mond [4] and in 1974 by Farr and Hanson [5]. Since then an extensive bibliography has been produced. For more information and literature on continuous-time problems, I cite [6,7] and the references cited therein, for smooth problems. For nonsmooth problems, I cite [8–10]. Multiobjective programs were studied, for example, in [11–15].

The applicability and importance of the concepts of convexity, generalized convexity, invexity and generalized invexity on getting optimality conditions and duality results in mathematical programming and optimization problems are well known. In the case of continuous-time problems, see, for instance, [6,10,12,13,16].

This paper will make use of the preinvexity notion. Preinvex functions were introduced in Hanson and Mond [17] as a generalization of non-differentiable invex functions. Preinvex functions can also be seen as functions which possess convex like properties over invex sets. Afterwards, Weir and Jeyakumar [18] and Weir and Mond [19] worked with preinvex functions in scalar-valued and vector-valued programs. L. Batista dos Santos et al. [20] defined preinvex functions in Banach spaces and obtained optimality conditions for vector abstract optimization problems.

In [21] a generalization of the Gordan's Theorem of the alternative (see [22]) in the continuous-time context is developed. Such theorem is valid for convex functions. In this work (in Section 3) a version of the Generalized Gordan's Theorem for preinvex functions is presented. The theorem is then used to prove other results in the subsequent sections.

In Section 4 the Geoffrion [23] scheme is applied for (VCP); i.e., the correlation between (VCP) and the associated weighting scalar problem is established.

Saddle point optimality for convex continuous-time problems is studied in [7,15,24,25]. Here (in Section 5) preinvex problems are considered.

Osuna-Gómez et al. [26] introduced a new dual problem for vector optimization problems with the special feature of being a scalar program. In this paper (see Section 6) a similar one for continuous-time problems is defined and duality theorems are proved.

Next some preliminaries are provided, that is, some notations and definitions common for the whole article.

2. Preliminaries

In this paper, all vectors are column vectors and a prime is used to denote transposition.

Let us denote by C the positive closed orthant of $\mathbb{R}^k$, so that given $u, v \in \mathbb{R}^k$, $u \geq v$ means $u - v \in C$ and $u > v$ means $u - v \in \operatorname{int}(C)$.

Let F be the set of all feasible solutions of problem (VCP) (which is supposed to be nonempty); i.e.,

$$F = \{x \in X : g(x(t), t) \leq 0 \text{ a.e. in } [0, T]\}.$$

Definition 2.1. A feasible solution $\bar{x}$ is said to be an *efficient solution* of (VCP) if there does not exist another feasible solution x such that $\phi(x) \leq \phi(\bar{x})$ and $\phi(x) \neq \phi(\bar{x})$.

Let

$$\phi_j(x) = \int_0^T f_j(x(t), t)\mathrm{d}t, \quad x \in X,\ j \in \{1, \ldots, p\},$$

where $f_j(x(t), t)$ denotes the j-th component of $f(x(t), t) \in \mathbb{R}^p$.

Definition 2.2. A feasible solution $\bar{x}$ is said to be a *properly efficient solution* of (VCP) if it is efficient and if there exists a scalar $M > 0$ such that, for each i, we have

$$\frac{\phi_i(\bar{x}) - \phi_i(x)}{\phi_j(x) - \phi_j(\bar{x})} \leq M$$

for some j such that $\phi_j(x) > \phi_j(\bar{x})$, when $x \in F$ and $\phi_i(x) < \phi_i(\bar{x})$.

Definition 2.3. Let S be a subset of a Banach space E. We say that S is *invex* with respect to $\eta : S \times S \to E$ if for all $x_1, x_2 \in S$ and for each $\lambda \in (0, 1)$,

$$x_2 + \lambda\eta(x_1, x_2) \in S.$$

Definition 2.4. If S is invex with respect to $\eta : S \times S \to \mathbb{R}^n$, a given function $\theta : S \to \mathbb{R}^k$ is called *preinvex* with respect to η if for all $x_1, x_2 \in S$ and for each $\lambda \in (0, 1)$,

$$\theta(x_2 + \lambda\eta(x_1, x_2)) \leq \lambda\theta(x_1) + (1 - \lambda)\theta(x_2).$$

3. Gordan's transposition theorem

A transposition theorem (or a theorem of the alternative) is an assertion about the solutions for a pair of systems of inequalities and/or equalities. These sort of theorems are widely used in establishing optimality criteria for programming problems. A good guide for theorems of the alternative in finite dimensions can be found in [22]. These theorems also have been generalized for arbitrary vector spaces, as can be seen, for example, in [27].

Gordan's Theorem is one of the transposition theorems. In [21] a generalization of Gordan's Theorem of the alternative in the continuous-time context is developed. Such theorem is valid for convex functions.

Here, we derive a continuous-time generalization of Gordan's Transposition Theorem for nonconvex functions. In fact, the theorem is valid for preinvex functions. Such theorem will be necessary in the next sections, where we will get optimality and duality results.

On establishing the result, we follow [7] and consider the scalar case of (VCP)

$$\begin{aligned} &\text{minimize} \quad \phi(x) = \int_0^T f(x(t), t)\mathrm{d}t \\ &\text{subject to} \quad g(x(t), t) \le 0 \quad \text{a.e. in } [0, T], \\ &\qquad\qquad\quad x \in X, \end{aligned} \tag{P}$$

and its dual problem

$$\begin{aligned} &\text{maximize} \quad \psi(u) \\ &\text{subject to} \quad u(t) \ge 0 \quad \text{a.e. in } [0, T], \end{aligned} \tag{D}$$

where

$$\psi(u) = \inf_{x \in X} \int_0^T [f(x(t), t) + u'(t) g(x(t), t)]\mathrm{d}t$$

with $f(x(t), t) = \xi(x)(t),\ \xi : X \to \Lambda_1^1[0, T]$, $g(x(t), t) = \gamma(x)(t),\ \gamma : X \to \Lambda_1^m[0, T]$ and $u \in L_\infty^m[0, T]$.

Zalmai defined the *perturbation function* $p : \Lambda_1^m[0, T] \to \mathbb{R}$ related to problem (P) as

$$p(y) = \inf_{x \in X} \left\{ \phi(x) = \int_0^T f(x(t), t)\mathrm{d}t : g(x(t), t) \le y(t) \text{ a.e. in } [0, T] \right\},$$

where $y(t)$ was called the *perturbation vector*. He also introduced the following stability concept for (P).

Problem (P) is called *stable* if $p(0) < \infty$ and there exists $M > 0$ such that

$$p(0) \le p(y) + M\|y\|_1 \quad \forall\, y \in \Lambda_1^m[0, T].$$

Remark. Concerning the invexity kernel of f and g below, we mean the map from $W \times W$ into $\mathbb{R}^n$ given by $\eta(x(t), y(t)) = \eta(x, y)(t),\ x, y \in X,\ t \in [0, T]$, where $W = \{x(t) \in \mathbb{R}^n : x \in X,\ t \in [0, T]\}$.

Lemma 3.1. *If X is an invex set with respect to $\eta : X \times X \to L_\infty^n[0, T]$ and $f(\cdot, t) = \xi(\cdot)(t)$ and $g(\cdot, t) = \gamma(\cdot)(t)$ a.e. in $[0, T]$ are preinvex in their first argument throughout $[0, T]$ with respect to the same η, then*

(i) *the feasible set of the perturbed problem $Y = \{y \in \Lambda_1^m[0, T] : g(x(t), t) \le y(t)$ a.e. in $[0, T]$ for some $x \in X\}$ is convex;*
(ii) *the perturbation function p is convex on Y.*

Proof. Let $y_1, y_2 \in Y$ and $\lambda \in (0, 1)$. Then there exist $x_i \in X$ such that $g(x_i(t), t) \le y_i(t)$ a.e. in $[0, T],\ i = 1, 2$. Since X is invex and g is preinvex with respect to η, there exists $x_2 + \lambda\eta(x_1, x_2) \in X$ such that

$$\begin{aligned} g(x_2(t) + \lambda\eta(x_1(t), x_2(t)), t) &\le \lambda g(x_1(t), t) + (1 - \lambda) g(x_2(t), t) \\ &\le \lambda y_1(t) + (1 - \lambda) y_2(t) \quad \text{a.e. in } [0, T]. \end{aligned}$$

Therefore $\lambda y_1 + (1 - \lambda) y_2 \in Y$.

Let $y_1, y_2 \in Y$ and $\lambda \in (0, 1)$. Let $\varepsilon > 0$. Then there exist $x_i \in X$ such that $g(x_i(t), t) \le y_i(t)$ a.e. in $[0, T]$ and $\phi(x_i) < p(y_i) + \varepsilon,\ i = 1, 2$. Since X is invex and g is preinvex with respect to η, we have $x_2 + \lambda\eta(x_1, x_2) \in X$ and

$$\begin{aligned} g(x_2(t) + \lambda\eta(x_1(t), x_2(t)), t) &\le \lambda g(x_1(t), t) + (1 - \lambda) g(x_2(t), t) \\ &\le \lambda y_1(t) + (1 - \lambda) y_2(t) \quad \text{a.e. in } [0, T]. \end{aligned}$$

Hence,

$$\begin{aligned} p(\lambda y_1 + (1 - \lambda) y_2) &\le \phi(x_2 + \lambda\eta(x_1, x_2)) \\ &\le \lambda\phi(x_1) + (1 - \lambda)\phi(x_2) \\ &< \lambda[p(y_1) + \varepsilon] + (1 - \lambda)[p(y_2) + \varepsilon] \\ &= \lambda p(y_1) + (1 - \lambda) p(y_2) + \varepsilon, \end{aligned}$$

where the second inequality above holds true provided f is preinvex with respect to η. Taking limits as $\varepsilon \downarrow 0$, it follows that p is convex on Y. □

Using this lemma and reasoning exactly as Zalmai did in [7] we get the strong duality theorem below.

Theorem 3.1. *If problem (P) is stable, then problem (D) has an optimal solution and the optimal values of (P) and (D) coincide.*

With this theorem in hand, we obtain, in a fully similar manner as Zalmai in [21], the partial generalization of Motzkin's and Slater's Transposition Theorems and the generalization of Gordan's Transposition Theorem for nonconvex functions. In what follows, such theorems are respectively presented.

Theorem 3.2. *Let $X \subset L^n_\infty[0,T]$, $g : \mathbb{R}^n \times [0,T] \to \mathbb{R}^m$ and $h : \mathbb{R}^n \times [0,T] \to \mathbb{R}^k$, where $g(x(t),t) = \gamma(x)(t)$ and $h(x(t),t) = \sigma(x)(t)$ a.e. in $[0,T]$, with $\gamma : X \to \Lambda^m_1[0,T]$ and $\sigma : X \to \Lambda^k_1[0,T]$. Assume that X is an invex set with respect to $\eta : X \times X \to L^n_\infty[0,T]$, g is preinvex in its first argument throughout $[0,T]$ with respect to the same η and that h is linear in its first argument throughout $[0,T]$. If there does not exist $x \in X$ such that*

$$g(x(t),t) < 0 \quad \text{a.e. in } [0,T],$$
$$h(x(t),t) = 0 \quad \text{a.e. in } [0,T],$$

then there exist $u \in L^m_\infty[0,T]$ and $v \in L^k_\infty[0,T]$ such that $u(t) \geq 0$, $(u(t), v(t)) \neq 0$ a.e. in $[0,T]$ and

$$\int_0^T [u'(t)g(x(t),t) + v'(t)h(x(t),t)]\mathrm{d}t \geq 0 \quad \forall\, x \in X.$$

Theorem 3.3. *Let $X \subset L^n_\infty[0,T]$, $g^j : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{m_j}$, $j = 1,2,3$, and $h : \mathbb{R}^n \times [0,T] \to \mathbb{R}^k$, where $g^j(x(t),t) = \gamma^j(x)(t)$, $j = 1,2,3$, and $h(x(t),t) = \sigma(x)(t)$ a.e. in $[0,T]$, with $\gamma^j : X \to \Lambda^{m_j}_1[0,T]$, $j = 1,2,3$, and $\sigma : X \to \Lambda^k_1[0,T]$. Assume that X is an invex set with respect to $\eta : X \times X \to L^n_\infty[0,T]$, g^j, $j = 1,2,3$, is preinvex in its first argument throughout $[0,T]$ with respect to the same η and that h is linear in its first argument throughout $[0,T]$. If there does not exist $x \in X$ such that*

$$g^1(x(t),t) < 0 \quad \text{a.e. in } [0,T],$$
$$g^2(x(t),t) \leq 0 \quad \text{a.e. in } [0,T],$$
$$g^2(x(t),t) \neq 0 \quad \text{a.e. in } [0,T],$$
$$g^3(x(t),t) \leq 0 \quad \text{a.e. in } [0,T],$$
$$h(x(t),t) = 0 \quad \text{a.e. in } [0,T],$$

then there exist $u^j \in L^{m_j}_\infty[0,T]$, $j = 1,2,3$, and $v \in L^k_\infty[0,T]$ such that $u^j(t) \geq 0$, $j = 1,2,3$, $(u^1(t), u^2(t), u^3(t), v(t)) \neq 0$ a.e. in $[0,T]$ and

$$\int_0^T [(u^1)'(t)g^1(x(t),t) + (u^2)'(t)g^2(x(t),t) + (u^3)'(t)g^3(x(t),t) + v'(t)h(x(t),t)]\mathrm{d}t \geq 0 \quad \forall\, x \in X.$$

Theorem 3.4 (*Generalized Gordan's Theorem*). *Let $X \subset L^n_\infty[0,T]$ and $g : \mathbb{R}^n \times [0,T] \to \mathbb{R}^m$, where $g(x(t),t) = \gamma(x)(t)$ a.e. in $[0,T]$ and $\gamma : X \to \Lambda^m_1[0,T]$. If X is an invex set with respect to $\eta : X \times X \to L^n_\infty[0,T]$ and g is preinvex in its first argument throughout $[0,T]$ with respect to the same η, then exactly one of the following systems is consistent:*

(I) *there exists $x \in X$ such that $g(x(t),t) < 0$ a.e. in $[0,T]$;*
(II) *there exists $u \in L^m_\infty[0,T]$, $u(t) \geq 0$, $u(t) \neq 0$ a.e. in $[0,T]$, such that $\int_0^T u'(t)g(x(t),t)\mathrm{d}t \geq 0\ \forall\, x \in X$.*

Pini [28] showed that differentiable preinvex functions are invex functions. Thereby, the theorems above are true for invex functions. In particular, we have an Invex Generalized Gordan's Theorem.

4. The scalarization method

A common strategy to characterize solutions of vector problems is the scalarization scheme: a scalar problem is associated to the multi-objective problem and from the solutions of the scalar problem, the solutions of the vector problem are characterized. Observe that we can, therefore, use all the known tools for scalar problems, both theoretical and numerical, in order to solve the vector one. But, we emphasize that to solve the scalarized problem it is not always equivalent to solving the vector one. Thus a key question is to know under what conditions we have the equivalence between them. Answering this question is of great importance, both theoretically and practically. There are some partial results to this question. See [23,26] for problems in finite dimensions and [29,30] for infinite dimensional problems. In [23], Geoffrion identifies the equivalence between the problems under convexity assumptions. In [26], Osuna-Gómez et al. obtain similar results by making use of generalized convexity. Analogous results are obtained in [29] for optimal control problems, and in [30] for infinite programming problems, both of them under differentiability hypothesis.

Thereafter we generalize the results cited above by making use of preinvexity assumptions and the Gordan's Theorem.

The scalarization method consists in converting the vector problem into a scalar problem, where the objective function is a weighted sum of the various objectives of the vector problem.

Let $V = \{v \in \mathbb{R}^p : v_1 + \cdots + v_p = 1 \text{ and } v > 0\}$.

Given $\nu \in V$, the weighting scalar problem associated with (VCP) is given as

$$\begin{aligned} &\text{minimize} \quad \Phi(x) = \sum_{j=1}^{p} \nu_j \phi_j(x) \\ &\text{subject to} \quad g(x(t), t) \leq 0 \quad \text{a.e. in } [0, T], \\ &\qquad\qquad\quad x \in X. \end{aligned} \tag{WSP$_\nu$}$$

A feasible solution $\bar{x}$ is an optimal solution of (WSP$_\nu$) if $\Phi(\bar{x}) \leq \Phi(x)$ for all $x \in F$.

The next two theorems show the connection between (VCP) and (WSP$_\nu$).

Theorem 4.1. *Assume that X is invex with respect to $\eta : X \times X \to L_\infty^n[0, T]$ and $f(\cdot, t) = \xi(\cdot)(t)$ and $g(\cdot, t) = \gamma(\cdot)(t)$ a.e. in $[0, T]$ are preinvex in their first argument throughout $[0, T]$ with respect to the same η. If $\bar{x} \in F$ is a properly efficient solution of (VCP), then there exists $\hat{\nu} \in V$, such that $\bar{x}$ is an optimal solution of* (WSP$_{\hat{\nu}}$).

Proof. Suppose that $\bar{x}$ is a properly efficient solution for (VCP). Then, for each $i \in \{1, \ldots, p\}$, the following system has no solution $x \in F$:

$$\begin{aligned} &\phi_i(x) < \phi_i(\bar{x}), \\ &\phi_i(x) + M\phi_j(x) < \phi_i(\bar{x}) + M\phi_j(\bar{x}), \quad j \neq i, \end{aligned}$$

where M is the constant of the definition of properly efficient solutions. Indeed, if there is a solution for this system, provided $\bar{x}$ is a properly efficient solution,

$$\frac{\phi_i(\bar{x}) - \phi_i(x)}{\phi_j(x) - \phi_j(\bar{x})} \leq M$$

for some j such that $\phi_j(x) > \phi_j(\bar{x})$. The set $\{j : \phi_j(x) > \phi_j(\bar{x})\}$ is nonempty, because otherwise

$$\phi_j(x) \leq \phi_j(\bar{x}) \quad \forall\, j \in \{1, \ldots, p\},$$

which would imply that $\bar{x}$ is not an efficient solution, since $\phi_i(x) < \phi_i(\bar{x}) \Rightarrow \phi(x) \neq \phi(\bar{x})$. Thus

$$\phi_i(x) + M\phi_j(x) \geq \phi_i(\bar{x}) + M\phi_j(\bar{x}),$$

which contradicts the second inequality of the system.

Set

$$\begin{aligned} &\varphi_i(x) = \phi_i(x) - \phi_i(\bar{x}), \\ &\varphi_j(x) = \phi_i(x) + M\phi_j(x) - \phi_i(\bar{x}) - M\phi_j(\bar{x}), \quad j \neq i. \end{aligned}$$

It is easy to see that F is invex with respect to η and φ_j, $j \in \{1, \ldots, p\}$, is preinvex with respect to η.

By the Generalized Gordan's Theorem 3.4, it follows that for each $i \in \{1, \ldots, p\}$ there exist $\nu^i \in \mathbb{R}^p$, $\nu^i \geq 0$, $\nu^i \neq 0$, such that

$$\sum_{j=1}^{p} \nu_j^i \varphi_j(x) \geq 0 \quad \forall\, x \in F.$$

Since $\nu^i \geq 0$ and $\nu^i \neq 0$, we can assume that $\nu_1^i + \cdots + \nu_p^i = 1$. So

$$\phi_i(x) + M \sum_{j \neq i} \nu_j^i \phi_j(x) \geq \phi_i(\bar{x}) + M \sum_{j \neq i} \nu_j^i \phi_j(\bar{x}) \quad \forall\, x \in F.$$

Summing over $i \in \{1, \ldots, p\}$ we obtain

$$\sum_{j=1}^{p} \tilde{\nu}_j \phi_j(x) \geq \sum_{j=1}^{p} \tilde{\nu}_j \phi_j(\bar{x}) \quad \forall\, x \in F,$$

where

$$\tilde{\nu}_j = 1 + M \sum_{i \neq j} \nu_j^i, \quad j \in \{1, \ldots, p\}.$$

Set $\hat{\nu} = \tilde{\nu}/(\tilde{\nu}_1 + \cdots + \tilde{\nu}_p) \in V$. Therefore $\bar{x}$ is an optimal solution of (WSP$_{\hat{\nu}}$). □

For the reciprocal no preinvexity assumption is necessary.

Theorem 4.2. *Let $\nu \in V$. If $\bar{x} \in F$ is an optimal solution of (WSP$_\nu$), then $\bar{x}$ is a properly efficient solution of (VCP).*

Proof. Let us suppose that there exists $x \in F$ such that $\phi(x) \le \phi(\bar{x})$ and $\phi(x) \neq \phi(\bar{x})$. Then, since $\nu > 0$,

$$\sum_{j=1}^{p} \nu_j \phi_j(x) < \sum_{j=1}^{p} \nu_j \phi_j(\bar{x}),$$

which contradicts the optimality of $\bar{x}$ in (WSP$_\nu$). Therefore $\bar{x}$ is an efficient solution of (WSP$_\nu$).

Let $M = (p-1)\max\{\nu_j/\nu_i : i, j = 1, \dots, p\},\ p \ge 2$. If there are $x \in F$ and $i \in \{1, \dots, p\}$, such that $\phi_i(x) < \phi_i(\bar{x})$ and

$$\frac{\phi_i(\bar{x}) - \phi_i(x)}{\phi_j(x) - \phi_j(\bar{x})} > M$$

for all j such that $\phi_j(x) > \phi_j(\bar{x})$, then

$$\phi_i(\bar{x}) - \phi_i(x) > \frac{p-1}{\nu_i} \nu_j [\phi_j(x) - \phi_j(\bar{x})] \quad \forall\, j \neq i.$$

It follows that

$$\frac{\nu_i}{p-1}[\phi_i(\bar{x}) - \phi_i(x)] > \nu_j[\phi_j(x) - \phi_j(\bar{x})] \quad \forall\, j \neq i.$$

Summing over $j \neq i$ we obtain

$$\nu_i[\phi_i(\bar{x}) - \phi_i(x)] > \sum_{j \neq i} \nu_j[\phi_j(x) - \phi_j(\bar{x})],$$

so that

$$\sum_{j=1}^{p} \nu_j[\phi_j(x) - \phi_j(\bar{x})] < 0,$$

which contradicts the optimality of $\bar{x}$ in (WSP$_\nu$). We conclude, then, that $\bar{x}$ is a properly efficient solution of (VCP). □

5. Saddle point optimality criteria

This section is devoted to saddle point optimality conditions. Essentially, an optimal solution of a constrained problem, under some conditions, is a saddle point of a unconstrained one, and vice-versa. In this kind of approach, no differentiability assumption is employed. In [22] this sort of study can be found for scalar mathematical programming problems and in [19,26] for multiobjective programming. On continuous-time problems, we can cite, for example, [7,15,25]. The saddle point definition which is given here is a generalization of the definition presented in [26]. As pointed out by the authors, in this definition the multiplier for the constraints is a vectorial and not a matrixial function. Furthermore, the vector saddle point conditions are scalar conditions, not vectorial ones.

We show that the problem of finding a properly efficient solution of (VCP) may be equivalent to solving the Kuhn–Tucker saddle point problem presented below. Indeed, the equivalence is obtained by making use of a constraint qualification and preinvexity hypothesis.

Let the scalar Lagrangian-type function $L : X \times V \times L_\infty^m[0, T] \to \mathbb{R}$ be defined by

$$L(x, v, u) = \int_0^T [v' f(x(t), t) + u'(t) g(x(t), t)]\mathrm{d}t.$$

A triple $(\bar{x}, \bar{v}, \bar{u}) \in X \times V \times L_\infty^m[0, T]$ is called a *vector Kuhn–Tucker saddle point* of (VCP) if $\bar{u}(t) \ge 0$ a.e. in $[0, T]$ and

$$L(\bar{x}, \bar{v}, u) \le L(\bar{x}, \bar{v}, \bar{u}) \le L(x, \bar{v}, \bar{u}) \tag{1}$$

for all $x \in X$ and all $u \in L_\infty^m[0, T],\ u(t) \ge 0$ a.e. in $[0, T]$.

Next we state a Slater-type constraint qualification in the continuous-time context.

We say that (VCP) satisfies (CQ) if there exists $\hat{x} \in X$ such that

$$g(\hat{x}(t), t) < 0 \quad \text{a.e. in } [0, T].$$

In the following results, we have necessary and sufficient conditions of saddle point optimality.

Theorem 5.1. *Assume that X is invex with respect to $\eta : X \times X \to L_\infty^n[0, T]$ and $f(\cdot, t) = \xi(\cdot)(t)$ and $g(\cdot, t) = \gamma(\cdot)(t)$ a.e. in $[0, T]$ are preinvex in their first argument throughout $[0, T]$ with respect to the same η. Further, assume that (VCP) satisfies (CQ). If $\bar{x}$ is a properly efficient solution, then there exist $\bar{v} \in V$ and $\bar{u} \in L_\infty^m[0, T]$ such that $\int_0^T \bar{u}'(t) g(\bar{x}(t), t)\mathrm{d}t = 0$ and $(\bar{x}, \bar{v}, \bar{u})$ is a vector Kuhn–Tucker saddle point of (VCP).*

Proof. Suppose that $\bar{x}$ is a properly efficient solution of (VCP). Then, reasoning as in the proof of Theorem 4.1 we see that, for each $i \in \{1, \ldots, p\}$, there is no solution $x \in X$ for the following system:

$$\begin{aligned}
&\phi_i(x) < \phi_i(\bar{x}),\\
&\phi_i(x) + Mf_j(x) < \phi_i(\bar{x}) + M\phi_j(\bar{x}), \quad j \neq i,\\
&g(x(t), t) < 0 \quad \text{a.e. in } [0, T],
\end{aligned}$$

where M is the constant of the definition of properly efficient solutions.

Setting

$$\begin{aligned}
&\varphi_i(x) = \phi_i(x) - \phi_i(\bar{x}),\\
&\varphi_j(x) = \phi_i(x) + M\phi_j(x) - \phi_i(\bar{x}) - M\phi_j(\bar{x}), \quad j \neq i,\\
&\chi(x(t), t) = g(x(t), t),
\end{aligned}$$

it follows from the Generalized Gordan's Theorem 3.4, that for each $i \in \{1, \ldots, p\}$ there exist $v^i \in \mathbb{R}^p$ and $u^i \in L_\infty^m[0, T]$ such that $(v^i, u^i(t)) \geq 0$, $(v^i, u^i(t)) \neq 0$ a.e. in $[0, T]$ and

$$\sum_{j=1}^{p} v_j^i \varphi_j(x) + \int_0^T (u^i)'(t)\chi(x(t), t)\mathrm{d}t \geq 0 \quad \forall\, x \in X.$$

If $v^i = 0$ we have

$$\int_0^T (u^i)'(t)\chi((x(t), t))\mathrm{d}t \geq 0 \quad \forall\, x \in X,$$

which is equivalent, by the Gordan's Theorem, to saying that there does not exist $x \in X$ such that $g(x(t), t) < 0$ a.e. in $[0, T]$, in disagreement with (CQ). Thus $v^i \neq 0$ and we can assume that $v_1^i + \cdots + v_p^i = 1$. So

$$\phi_i(x) + M\sum_{j\neq i} v_j^i \phi_j(x) + \int_0^T (u^i)'(t)g(x(t), t)\mathrm{d}t \geq \phi_i(\bar{x}) + M\sum_{j\neq i} v_j^i \phi_j(\bar{x}) \quad \forall\, x \in X.$$

Summing over $i \in \{1, \ldots, p\}$ we obtain

$$\sum_{j=1}^{p} \tilde{v}_j \phi_j(x) + \int_0^T \tilde{u}'(t)g(x(t), t)\mathrm{d}t \geq \sum_{j=1}^{p} \tilde{v}_j \phi_j(\bar{x}) \quad \forall\, x \in X, \tag{2}$$

where

$$\tilde{v}_j = 1 + M\sum_{i\neq j} v_j^i, \quad j \in \{1, \ldots, p\}$$

and

$$\tilde{u}(t) = \sum_{i=1}^{p} u^i(t), \quad t \in [0, T].$$

Using $x = \bar{x}$ in (2) we get

$$\int_0^T \tilde{u}'(t)g(\bar{x}(t), t)\mathrm{d}t \geq 0.$$

But clearly $\tilde{u}(t) \geq 0$ a.e. in $[0, T]$, and $g(\bar{x}(t), t) \leq 0$ a.e. in $[0, T]$, provided $\bar{x} \in F$, so that we get the opposite inequality above. Then

$$\int_0^T \tilde{u}'(t)g(\bar{x}(t), t)\mathrm{d}t = 0. \tag{3}$$

From (2) and (3) we have

$$\begin{aligned}
\int_0^T [\bar{v}'f(x(t), t) + \bar{u}'(t)g(x(t), t)]\mathrm{d}t &\geq \int_0^T [\bar{v}'f(\bar{x}(t), t) + \bar{u}'(t)g(\bar{x}(t), t)]\mathrm{d}t\\
&= \int_0^T \bar{v}'f(\bar{x}(t), t)\mathrm{d}t\\
&\geq \int_0^T [\bar{v}'f(\bar{x}(t), t) + u'(t)g(\bar{x}(t), t)]\mathrm{d}t
\end{aligned}$$

for all $x \in X$ and all $u \in L^m_\infty[0,T]$, $u(t) \geq 0$ a.e. in $[0,T]$, where $\bar{v} = \tilde{v}/(\tilde{v}_1 + \cdots + \tilde{v}_p)$ and $\bar{u} = \tilde{u}/(\tilde{v}_1 + \cdots + \tilde{v}_p)$. Therefore $(\bar{x}, \bar{v}, \bar{u}) \in X \times V \times L^m_\infty[0,T]$ is a vector Kuhn–Tucker saddle point of (VCP) with $\int_0^T \bar{u}'(t) g(\bar{x}(t), t)\mathrm{d}t = 0$. □

The converse is always valid.

Theorem 5.2. *If $(\bar{x}, \bar{v}, \bar{u})$ is a vector Kuhn–Tucker saddle point, then $\bar{x}$ is a properly efficient solution of (VCP).*

Proof. Setting $u \equiv 0$ in (1) it comes

$$\int_0^T \bar{v}' f(\bar{x}(t), t)\mathrm{d}t \leq \int_0^T [\bar{v}' f(x(t), t) + \bar{u}'(t) g(x(t), t)]\mathrm{d}t$$

for all $x \in X$. Particularly for all $x \in F$. In such case,

$$\int_0^T [\bar{v}' f(x(t), t) + \bar{u}'(t) g(x(t), t)]\mathrm{d}t \leq \int_0^T \bar{v}' f(x(t), t)\mathrm{d}t,$$

so that

$$\int_0^T \bar{v}' f(\bar{x}(t), t)\mathrm{d}t \leq \int_0^T \bar{v}' f(x(t), t) \quad \forall\, x \in F,$$

i.e., $\bar{x}$ is optimal in $(WSP_{\bar{v}})$. From Theorem 4.2 we know that $\bar{x}$ is a properly efficient solution of (VCP). □

6. The scalar dual problem

In general, the dual of a vector programming problem is also a vector problem. See, for example, [26,31] for mathematical programming problems. Nobakhtian and Pouryayevali [13] studied Wolfe and Mond–Weir type dual problems for continuous-time programs. Zalmai [15,25] still considered one Lagrangian type dual. Nevertheless, in [26] is presented a formulation of the Lagrangian dual problem which is characterized as being a scalar program. This model is followed in this section. The relationship between (VCP) and the scalar Lagrangian-type dual problem defined below is studied, through the weak and strong duality theorems.

$$\begin{aligned} &\text{Maximize} \quad \psi(v, u) \\ &\text{subject to} \quad u(t) \geq 0 \quad \text{a.e. in } [0,T], \end{aligned} \tag{SDP}$$

where $\psi : V \times L^m_\infty[0,T] \to \mathbb{R}$ is given by

$$\psi(v, u) = \inf_{x \in X} L(x, v, u) = \inf_{x \in X} \int_0^T [v' f(x(t), t) + u'(t) g(x(t), t)]\mathrm{d}t.$$

Let $\bar{v} \in V$. A feasible solution $\bar{u}$ is an optimal solution of (SDP) if $\psi(\bar{v}, \bar{u}) \geq \psi(\bar{v}, u)$ for all $u \in L^m_\infty[0,T]$, $u(t) \geq 0$ a.e. in $[0,T]$.

In the sequel, the duality theorems are stated.

Theorem 6.1 (*Weak Duality*). *Let $\hat{x}$ and $\hat{u}$ be feasible solutions of (VCP) and (SDP), respectively, and $\hat{v} \in V$. Then $\hat{v}' \phi(\hat{x}) \geq \psi(\hat{v}, \hat{u})$.*

Proof. Being $\hat{x}$ and $\hat{u}$ feasible solutions of (VCP) and (SDP), we have $g(\hat{x}(t), t) \leq 0$ and $\hat{u}(t) \geq 0$ a.e. in $[0,T]$, so that $\hat{u}'(t) g(\hat{x}(t), t) \leq 0$ a.e. in $[0,T]$. Thus

$$\hat{v}' \phi(\hat{x}) \geq \hat{v}' \phi(\hat{x}) + \int_0^T \hat{u}'(t) g(\hat{x}(t), t)\mathrm{d}t = L(\hat{x}, \hat{v}, \hat{u}) \leq \inf_{x \in X} L(x, \hat{v}, \hat{u}) = \psi(\hat{v}, \hat{u}). \quad \square$$

Corollary 6.1. *Let $\bar{x}$ and $\bar{u}$ be feasible solutions for (VCP) and (SDP), respectively, and $\bar{v} \in V$, be such that $\bar{v}' \phi(\bar{x}) \leq \psi(\bar{v}, \bar{u})$. Then $\bar{x}$ is a properly efficient solution of (VCP) and $\bar{u}$ is an optimal solution of (SDP).*

Proof. By the Weak Duality Theorem 6.1 we have that

$$\bar{v}' \phi(\bar{x}) \geq \psi(\bar{v}, u) \quad \forall\, u \in L^m_\infty[0,T], \qquad u(t) \geq 0 \quad \text{a.e. in } [0,T].$$

In this way, $\psi(\bar{v}, u) \leq \bar{v}' \phi(\bar{x}) \leq \psi(\bar{v}, \bar{u})$ $\forall\, u \in L^m_\infty[0,T]$, $u(t) \geq 0$ a.e. in $[0,T]$. Therefore $\bar{u}$ is an optimal solution of (SDP).

By the Weak Duality Theorem 6.1 we have also that

$$\bar{v}' \phi(x) \geq \psi(\bar{v}, \bar{u}) \quad \forall\, x \in F.$$

Then $\bar{v}' \phi(x) \geq \psi(\bar{v}, \bar{u}) \geq \bar{v}' \phi(\bar{x})$ $\forall\, x \in F$. Consequently $\bar{x}$ is an optimal solution for the weighting scalar problem $(WSP_{\bar{v}})$. By Theorem 4.2, $\bar{v}$ is a properly efficient solution for (VCP). □

Theorem 6.2 (*Strong Duality*). *Assume that X is invex with respect to $\eta : X \times X \to L^n_\infty[0, T]$ and $f(\cdot, t) = \xi(\cdot)(t)$ and $g(\cdot, t) = \gamma(\cdot)(t)$ a.e. in $[0, T]$ are preinvex in their first argument throughout $[0, T]$ with respect to the same η. Further, assume that (VCP) satisfies (CQ). If $\bar{x}$ is a properly efficient solution of (VCP), then there exist $(\bar{v}, \bar{u}) \in V \times L^m_\infty[0, T]$, $\bar{u}(t) \geq 0$ a.e. in $[0, T]$, such that $\bar{u}$ is an optimal solution of (SDP) and $\bar{v}'\phi(\bar{x}) = \psi(\bar{v}, \bar{u})$.*

Proof. It is clear that the assumptions of Theorem 5.1 are satisfied. Provided $\bar{x}$ is a properly efficient solution, there exist $\bar{v} \in V$ and $\bar{u} \in L^m_\infty[0, T]$ such that $\int_0^T \bar{u}'(t)g(\bar{x}(t), t)\mathrm{d}t = 0$ and $(\bar{x}, \bar{v}, \bar{u})$ is a vector Kuhn–Tucker saddle point for (VCP), so that (1) holds. Let u be a feasible solution for (SDP). We have that,

$$\psi(\bar{v}, u) = \inf_{x \in X} L(x, \bar{v}, u) \leq L(\bar{x}, \bar{v}, u) \leq L(\bar{x}, \bar{v}, \bar{u}),$$

where the last inequality follows from (1). From (1) it follows also that $L(\bar{x}, \bar{v}, \bar{u}) \leq L(x, \bar{v}, \bar{u})$ for all $x \in X$. Thus, since $\bar{x} \in X$, $\psi(\bar{v}, \bar{u}) = L(\bar{x}, \bar{v}, \bar{u})$. So, $\psi(\bar{v}, u) \leq \psi(\bar{v}, \bar{u})$ for all feasible solution u, that is, $\bar{u}$ is an optimal solution for (SDP). Furthermore,

$$\psi(\bar{v}, \bar{u}) = L(\bar{x}, \bar{v}, \bar{u}) = \int_0^T [\bar{v}'f(\bar{x}(t), t) + \bar{u}'(t)g(\bar{x}(t), t)]\mathrm{d}t = \bar{v}'\phi(\bar{x}),$$

since $\int_0^T \bar{u}'(t)g(\bar{x}(t), t)\mathrm{d}t = 0$, as we know from Theorem 5.1. □

7. Final considerations

This paper dealt with continuous-time programming problems with multiple objectives. The results were formulated without using differentiability. The preinvexity concept and the Gordan's Transposition Theorem (which was generalized for preinvex functions) were used for establishing the most results.

Saddle point type optimality conditions were presented, where the inequalities were scalar ones, since a scalar Langrangian was utilized. The same Lagrangian was employed in the definition of a scalar dual problem. Necessary and sufficient saddle point optimality conditions were given and the weak and strong duality theorems were stated. Moreover, optimality conditions were achieved by means of the scalarization method, also known as Geoffrion's scheme.

Acknowledgement

The author gratefully acknowledges the financial support of State of Minas Gerais Research Foundation – FAPEMIG – under the grant CEX – APQ-00686-09.

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